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ONE APPROACH TO FIND OPTIMAL CONTROLS FOR DISCRETE DYNAMIC SYSTEMS WITH NUMERICAL METHODS APPLICATION

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Abstract. It is proposed the approach of numerical methods application to find optimal controls for discrete dynamic systems on the base of reduction to solving the corresponded problem about optimisation of some several variable function with some restriction, so that this proposed approach is suitable both for linear and both for nonlinear systems. The principal idea is in representing the control through the finite number of scalar parameters and in building the function of these parameters, so that the optimum of this function will represent the optimal control approximately. Although the proposed approach leads to the approximate solutions and requires a lot of computations, but this approach is suitable for solving the applied tasks, and finding of the optimal controls for the wheeled electromechanical platform is considcered as the example of using this approach.

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1 Introduction

Improvement of the control gives a lot of possibilities to increase the operational efficiency even without the significant changes in building of systems, and it is widely used at present for different applications. It is naturally, that the optimality is the key property providing control improvement for different systems, so that considering of the fundamental problems about the optimal control, like in the research Yakub et al. (2021) for example, is pincipally required to solve different tasks in engineering (Kim & Singh, 2022), biology (Day & Taylor, 2000), public health (Aghdaoui et al., 2021) and others fields. Thus, the proposed research is in current interest because of considering the universal approach, based on the numerical methods to directly find the optimal controls for the general case of the discrete dynamic systems, and it is suitable for different applications.

The most fundamental of existed results in the field of optimal control is the Pontryagin's maximum principle developed in 1960-th, so that a lot of researchers use the ideas of this principle to formulate new theoretical results like in research (Mahmudov, 2021), for example, in which some sufficient optimality condition was derived. The research Cardin & Spiro (2019)

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shows us the new insightful way to derive the Pontryagin's maximum principle, and it suggests generalizations in diverse directions of such famous principle. The research Day & Taylor (2000) deals with generalisation of the Pontryagin's maximum principle to consider the dynamic evolutionary games between genetically related individuals, and the corresponded theorems was considered.

Using of Pontryagin's maximum principle allows considering also the optimal control problem for different engineering tasks. Although, the Pontryagin's maximum principle is the fundamental result on the theory of control, but it gives the way to solve a lot of particular tasks for different applications. This principle was used in the researches Lohneac et al. (2022); Pereira (2021); Song et al. (2020); Ritter et al. (2022) to find control optimal in time, in energy and in others sences for different kinds of engineering systems, as well as to find optimal control of different pandemic processes, like in the researches Aghdaoui et al. (2021); Riouali et al. (2022) for example. The importance of the Pontryagin's maximum principle is significant, but using it to solve the particular tasks requires understanding a lot of mathematical notions, and it can be realised only for separate classes of the considered systems, so that using of computers is necessary only to make the correspondent calculations, wich must be epspecially biult for each particular task. Thus, problems about an optimal control for dynamic systems are in considerations more than 50 years yet, but the most of the existed principal results are in the field of pure mathematics, so that we have no suitable universal approaches to direct solve tasks about optimal control for different applications using numerical methods at present. At the same time, we have a lot of numerical methods to direct solve the ordinary differential equations representing the mathematical models of different discrete systems, and, due to it, we have wide opportunities in computer simulations for different, including complicated, systems. For example, the complicated systems with interacted mechanical, electrical and electronic components was considered both for computer simulations and both for indirect measures in the research Mamalis et al. (2021), and a lot of others similar researches are existed. It is naturally, that wide opportunities in computer simulations of complicated systems allow considering a lot of applications, but limited oppportunities in using the numerical methods lead to necessities of the pure mathematics methods using to solve the optimal control problems, and it makes difficult considering a lot of correspondent applications in the fields of controls improvements. The principal difficulties in using the numerical methods to direct find the optimal controls are due to involving the abstract mathematical notions like the permissible controls set and others similars required for correspondent problems, but not agreeded with the discrete nature of numerical methods.

Optimisation for different applications, discussed in the researchers Corriou (2021); Ravindran et al. (2006); Yang (2018) and others similars, is the relatively separate field significantly based on the numerical methods. Such numerical optimization envisages considering the problem about the minimum or maximum value of some several variables function representing the used optimality criteria, and a lot of numerical methods was proposed to solve it, including with only computing the value of the researched function for its given arguments. Taking into account this circumstance, it seems naturally to reduce the optimal control problem to equivalent problem about minimization or maximization of some several variables function and to use the well-known numerical methods described in the researches Corriou (2021); Ravindran et al. (2006); Yang (2018) to solve this equivalent problem. Such approach will allow directly finding the optimal controls for the dynamic systems by using the corresponding numerical methods.

Taking into account all noted above circumstances, the principal purpose of this research is in development of the approaches for numerical methods using suitable for the different applications to find the optimal controls for discrete dynamic systems. To realise this principal purpose, the following particular objectives will be accomplished:

- it will be considered the general mathematical formulation of the optimal control problem for the common case of discrete dynamic systems;

- it will be developed the approach for reducing of the formulated optimal control problem to the resolving problem about minimization of some several variables function to use further the numerical methods to direct find of the optimal cointrol;
- it will be considered the particular example of using the proposed approach to find the optimal control for some given electromechanical wheeled platform.

These particular objectives really allow us to fully accomplish of the formulated principal purpose, although it is difficult to envisage all possible difficulties through one considered example only.

2 Mathematical formulation of the problem

We will consider further the optimal controls only for the discrete dynamic systems with the mathematical models represented by the finite systems of ordinary differential equations and the correspondent initial conditions. The mathematical formulation of the optimal control problem will be presented in the view enough only to use the numerical methods further.

Optimal control problem envisages, that the state of the researched system is changed during the time starting thom some given initial time moment, so that we will imagine the time as the continuous variable with the values $t \geq t_0$, where t_0 is the time value corresponded to the initial time moment. To represent the mathematical formulation of the problem we will introduce the time dependent finite-dimensional vector $\mathbf{x} = \mathbf{x}(t)$ representing the state of the researched discrete dynamic system. Besides, to consider the optimal controls we must introduce also the time dependent finite-dimensional control vector $\mathbf{u} = \mathbf{u}(t)$. We will assume that the mathematical model of the researched discrete dynamic system will have the following view with taking into account the introduced above notions:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}; \mathbf{u}), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{1}$$

where $\mathbf{f}$ is the given vector function defining the differential equations, and $\mathbf{x}_0$ is the given vector defining the state of the researched system at the given initial time moment $t = t_0$.

It is naturally, that the controls $\mathbf{u} = \mathbf{u}(t)$ cannot be arbitrary and must be limited in agreement with the considered problem and it mathematical model (1). Such limitations of the controls, at least, must provide existance of the solution for the initial-value problem (1). Besides, such limitations can involve some additional restrictions borned from the sense of the considered problem, so that for the most applications they can be generally represented in the form:

$$\|\mathbf{u}(t)\| \leq C_{\mathbf{u}}, C_{\mathbf{u}} > 0 \quad \forall t \geq t_0 \tag{2}$$

where $\|\bullet\|$ is the norm in some suitable functional space and $C_{\mathbf{u}} < \infty$ is some given value.

Let introduce the set U of the controls providing existance of the solution for the initial-value problem (1) and satisfying the limitation (2), whose represents usually the possible used power in the case of technical systems. Mathematical model (1) allows defining the state of the researched discrete dynamic system as the parameteric set of time depended functions $\mathbf{x} = \mathbf{x}(t)$ with the parameter corresponded to the control $\mathbf{u} \in U$. So, to show that the time dependent state $\mathbf{x} = \mathbf{x}(t)$, defined through the mathematical model (1), is actually corresponded to some control $\mathbf{u} \in U$, we will use the follow relation:

$$\mathbf{x} = \mathbf{x}(t; \mathbf{u}). \tag{3}$$

The relation (3) actually represents the solutions of the initial-value problem (1), so we cannot have this relation in analytical view in a general case. At the same time, the initial-value problem (1), representing the mathematical model of the researched discrete dynamic system, can be approximately solved by using the different well-known numerical methods (Korn & Korn, 2000) for each given control $\mathbf{u} = \mathbf{u}(t)$ satisfying the necessary conditions, so that $\mathbf{u} \in U$.

The controls $\mathbf{u} = \mathbf{u}(t)$ are actually required to provide optimal transitions of the researched dynamic system (1) from some given initial states to some given wished final states, so that the principal problem is in finding of the optimal controls $\mathbf{u} = \mathbf{u}(t)$, proviiding such optimal transitions, but not in modelling of the researched dynamic system (1) for some given controls $\mathbf{u} = \mathbf{u}(t)$. Actually, it is not necessary, and, even, it is not possible practically to provide controlling and governing for all the state parameters of the researched dynamic system (1) included in it state vector $\mathbf{x} = \mathbf{x}(t)$. It is principally necessary to provide controlling and governing only for some parameters, which will be named further as the controlled parameters, and which will be imagined as the components of the time depended vector denoted as $\mathbf{y} = \mathbf{y}(t)$ and named as the vector of controlled parameters of the researched dynamic system. The vector $\mathbf{y}$ of controlled parameters actually gives some representation of the state of the researched dynamic system (1) defined by the state vector $\mathbf{x}$, so that the following relation is existed:

$$\mathbf{y} = \mathbf{y}(\mathbf{x}), \tag{4}$$

where $\mathbf{y}(\mathbf{x})$ is some given function giving the definition of the controlled parameters of the researched dynamic system.

The relation (4) shows, that time dependence of the control parameters $\mathbf{y}(t)$ cannot be arbitrary, and it is predefined only by changes of the state of researched system (1) represented by time dependence of the state vector $\mathbf{x}(t)$. Due to the relation (4), we can see, that the vector $\mathbf{y}$ of controlled parameters at the initial time moment $t = t_0$ can be defined by using the initial condition form the mathematical model (1) for the state vector $\mathbf{x}$:

$$\mathbf{y}(t_0) = \mathbf{y}_0, \quad \mathbf{y}_0 = \mathbf{y}(\mathbf{x}_0). \tag{5}$$

Taking into account the relation (4), defining the controlled parameters, and the relation (3), representing the state of the researched dynamic system (1) for the given control, we will see, that the vector of the controlled parameters will be depended on the time t in correspondence with the each of particular control $\mathbf{u}(t)$, and, similarilly with the view (3), we will have the following:

$$\mathbf{y} = \mathbf{y}(t; \mathbf{u}). \tag{6}$$

The control $\mathbf{u} = \mathbf{u}(t)$ is actually required to provide transition of the system (1) from given initial state, defined by the value t_0 and the state vector $\mathbf{y}_0$, to some given wished final state, defined by the time value t_f and the given state vector $\mathbf{y}_f$, so that this wished final state must satisfy the following relation:

$$\mathbf{y}(t_f) = \mathbf{y}_f. \tag{7}$$

Relations (6) and (7) allow defining the set $U_f \subset U$ of the permissible controls providing transition of the considered system (1) from the initial state to the final state in the following view:

$$U_f : \forall \mathbf{u} \in U_f \Rightarrow \mathbf{y}(t_f; \mathbf{u}) \equiv \mathbf{y}_f. \tag{8}$$

The relation (8) defines a lot of different controls in the general case, so to have the optimal control, we must define the optimality criteria, and such criteria is defined ussually by means some functional:

$$J = J(\mathbf{u}(t)), \quad J(\mathbf{u}(t)) = \int_{t_0}^{t_f} g(t, \mathbf{x}(t, \mathbf{u}(t)), \mathbf{u}(t))\, dt, \tag{9}$$

where $g(t, \mathbf{x}, \mathbf{u})$ is some given function defining the sence of the control optimality.

The functional (9) allows defining the optimal control:

$$\mathbf{u}_{\text{opt}}(t) \in U_f : \quad J(\mathbf{u}_{\text{opt}}(t)) \leq J(\mathbf{u}(t)) \quad \forall \mathbf{u}(t) \in U_f, \tag{10}$$

where $\mathbf{u}_{\text{opt}}(t)$ is the optimnal control.

It is understandable, that existance of the optimal control problem solution, defined by the relation (10), will be possible only due to the correspondent properites of the researched system mathematical model (1), of the controlled parameters definition (4), of the final state (7), as well as of the function (9) defining the optimality condition. We will not discuss futher the thoretical aspects of the optimal control problem, but we will develope only the approach to use the numerical methods to find the optimal control.

3 Approach to solution based on numerical methods

Difficulties in using the numerical methods to find the optimal control, defined by the relation (10), is due to the complicated nature of the set (8) and the the functional (9), whose are defined indirectly through the soplution of the nonlinear initial-value problem (1).

To compute the functional (9) value for the given control $\mathbf{u}(t)$ by using the numerical methods, it is suitable to represent the integral from the optimality criteria (9) in the equivalent form of the ordinary differential equation and the initial condition:

$$\frac{dJ}{dt} = g(t, \mathbf{x}, \mathbf{u}), \quad J(t_0) = 0. \tag{11}$$

The differential equation with the initial condition (11) can be considered in couple with the differential equations and initial conditions (1), so that computation of the functional (9), defining the optimality criteria, can be reduced to the solution of the initial-value problem, and it is possible to use different well-known numerical methods, like Runge-Kutte and others (Korn & Korn, 2000), to do it. Thus, computation of the functional value (9), defining the optimaliity criteria, has some difficulties, because it requires solving the nonlinerar initial-value problem (1), (11), but these difficulties are not principal due to using the corresponded well-known numerical methods.

The difficulties in using of the control set U_f definition through the relation (10) to find the optimal control (10) numerically are because of the numerical methods have the discrete nature counterwise with the set U_f inherent properties. So, to use numerical methods to find the optimal control (10), it is necessary to represent the set U_f in the suitable discrete form, and, to do it, we will assume, that all of the controls $\mathbf{u}(t)$ can be defined through the finite number n of the some scalar parameters $a_1, a_2, \ldots, a_n$. This assumption can be equivalently represented by means the following relation

$$\mathbf{u}(t) = \tilde{\mathbf{u}}(t; a_1, a_2, \ldots, a_n), \tag{12}$$

where $\tilde{\mathbf{u}}(t; a_1, a_2, \ldots, a_n)$ is some given function.

The control representation (12) allow us to reduce finding of the optimal control (10) to defining of the correspondent scalar parameters $a_k, k = 1, 2, \ldots, n$. To do it, we must reformulate the initial-value problem (1), (11) taking into account the control representation (12), so it will lead to the following:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}; a_1, a_2, \ldots, a_n), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{13}$$

$$\frac{dJ}{dt} = g(t, \mathbf{x}; a_1, a_2, \ldots, a_n), \quad J(t_0) = 0, \tag{14}$$

where $\mathbf{f}(t, \mathbf{x}; a_1, a_2, \ldots, a_n)$ is the vector function defined by substitution of the control representation (12) to the the introduced above vector function $\mathbf{f}(t, \mathbf{x}; \mathbf{u})$; $g(t, \mathbf{x}; a_1, a_2, \ldots, a_n)$ is the function defined by substitution of the control representation (12) to the the introduced above function $g(t, \mathbf{x}; \mathbf{u})$.

It is understansdable, that solution of the initial-value problem (13), (14) will be the time dependent vector $\mathbf{x}(t)$ corresponded to each particular set of the parameters $a_1, a_2, \ldots, a_n$, defining the control $\mathbf{u}(t)$ in the agreement with the relation (12), and, similarilly to the view (3), we will represent such cerrespondence as following:

$$\mathbf{x} = \mathbf{x}(t; a_2, a_2, \ldots, a_n), \tag{15}$$

where $\mathbf{x}(t; a_2, a_2, \ldots, a_n)$ is actually the introduced above solution $\mathbf{x}(t; \mathbf{u})$ corresponded to the control (12).

The solution of the initial-value problem (13), (14), represented in the view (15), allows transforming the controlled parameters (4) and the functional (9), defining the optimality criteria (9), to the following:

$$\mathbf{y} = \mathbf{y}(t; a_2, a_2, \ldots, a_n), \quad J = J(t; a_2, a_2, \ldots, a_n), \tag{16}$$

where $\mathbf{y}(t; a_2, a_2, \ldots, a_n)$ is actually the introduced above controlled parameters $\mathbf{y}(t; \mathbf{u})$, and $J(t; a_2, a_2, \ldots, a_n)$ is the function representing the introduced above functional (9), defining the optimality criteria, so that all of them correspond to the control (12) through the inital-value problem (13), (14).

The condition (8), defining the set U_f of the permissible controls providing achievement of the given final state from the given initial state in agreement with the mathematical model (1), must be reformulated regarding with the used control representation (12) in the term of the scalar parameters $a_k, k = 1, 2, \ldots, n$. So, taking into account the conditions (2) and (8) instead the set U_f, we will have the set A_f^n of the sets $\{a_1, a_2, \ldots, a_n\}$ of the scalar parameters $a_k, k = 1, 2, \ldots, n$, providing achievement of the given final state from the given initial state in agreement with the mathematical model (13)-(16), in the following view:

$$A_f^n : \forall \{a_k\}_{k=1}^n \in A_f^n \Rightarrow \mathbf{y}(t_f; a_1, a_2, \ldots, a_n) \equiv \mathbf{y}_f \bigvee \|\tilde{\mathbf{u}}(t; a_1, a_2, \ldots, a_n)\| \le C_{\mathbf{u}}, \tag{17}$$

where $\left\{a_k^{\mathrm{opt}}\right\}_{k=1}^n$ is notation used for compact representation of the set $\{a_1, a_2, \ldots, a_n\}$.

Taking into account the definition (17) of the set A_f^n and second relation (14), we can reduce finding of the optimal control, represented by the condition (10), to finding the following optimal values:

$$\left\{a_k^{\mathrm{opt}}\right\}_{k=1}^n \in A_f^n : \quad J\left(t_f; a_1^{\mathrm{opt}}, a_2^{\mathrm{opt}}, \ldots, a_n^{\mathrm{opt}}\right) \le J(t_f; a_1, a_2, \ldots, a_n) \quad \forall \{a_k\}_{k=1}^n \in A_f^n. \tag{18}$$

The set $\left\{a_k^{\mathrm{opt}}\right\}_{k=1}^n$, defined by the condition (18), and the representation (12) will allow us to have the approximate representation of the optimal control:

$$\mathbf{u}_{\mathrm{opt}}(t) \approx \tilde{\mathbf{u}}\left(t; a_1^{\mathrm{opt}}, a_2^{\mathrm{opt}}, \ldots, a_n^{\mathrm{opt}}\right). \tag{19}$$

Due to the approximate relation (19), we can see, that finding of the optimal control (10) is reduced to finding of the minimum value (18) of the several variable function, defined by second relation in (15), inside the domain A_f^n, defined by the condition (17). To find the minimum of this several variables function J in the domain A_f^n, it is possible to use the well-known numerical optimisation methods (Corriou, 2021; Ravindran et al., 2006; Yang, 2018) requiring only computation of the values of the minimised function J for the given values of its arguments $a_1, a_2, \ldots, a_n$. At the same time, using the numerical optimisation methods to minimise the function J in the domain A_f^n, we will have difficulties due to this function and this domain are defined indirectly through the initial-value problem (13). Although, a lot of well-known numerical methods can be used to solve the initial-value problem (13), but necessity of solving this initial-value problem to compute the value of minimised function J will lead to significant computing time to find of the minimum value of such indirect defined function J, especially in the case of big number n of it argiments $a_1, a_2, \ldots, a_n$.

It is necessary to note, that the control representation (12) significantly limits the class of the considered controls, and it is really difficult to establish the correspondence degree between the exact solution (10) and the approximate numerical solution used in the relation (19). Thus, it is impossible to expect, that the approximate numerical solution, used in the approximate relation (19), will represent the exact solution (10) closely enough in general case, so that the proposed approach of using the numerical methods to find the optimal control will not allow having any reliable imaginations about the exact solution for the optimal control defined by the relation (10). Nevertheless, although, the proposed approach to find the optimal control numerically cannot claim on general resolving the optimal control problem, but this approach is important for different applications. Really, it is actually impossible to realise all the controls from the set U_f for the existed controlled systems due to the inherent limitations in kinematics, in power and others, so that we have the limited possible controls for all really existed systems, and such limited controls for a lot of really existed systems can be represented by the relation:

$$\mathbf{u}\left(t\right) = \sum_{k=1}^{n} a_k \boldsymbol{\varphi}_k\left(t\right), \tag{20}$$

where $\boldsymbol{\varphi}_k\left(t\right)$ are some given vector functions with the dimensions equal to the dimension of the control vector $\mathbf{u}$, and which represent the possible control modes of the researched system.

The relation (20), representing the possible controls in the really existed, but not in thoretically imagined systems, is actually the particular case of the general relation (12) representing the control through finite numbet of scalar parameters. Due to it, the proposed approach, based on the general relation (12), is really suitable to find optimal controls in really existed systems, representing different applications, although, this approach cannot solve the optimal control problem in general sence (10).

4 Example of the proposed approaches application

The system represented by the significantly nonlinear ordinary differential equations, so that linearisation is not possible principally, will be used to illustrate the proposed approaches for numerical methods application to find the optimal control for the discrete dynamic systems. Using of the significantly nonlinear system as the example will allow showing of exactly numerical methods usefulness.

The electromechanical six-wheeled platform fig. 1 is considered as the example of the researched discrete dynamic system. The straight-line motion of the housing-1 of the considered six-wheeled platform (fig. 1a) is due to rotations of the wheels-2, and rotations of the wheels-2 are due to the electromechanical couples M_{e}, which are provided by the driving direct current electric motors-3, so that one separate motor is envisaged for each separate wheel. We will assume, that all of the platform wheels are rotated without sliding, so the rotation angle q_1 of the wheels fully defines the state and can be considered as the generalised coordinate representing all the mechanical parts of the researched platform (fig. 1a). The significant nonlinearities in the differential equations, representing the mathematical model for such electromechanical wheeled platform (fig. 1a), are due to the aerodynamic damping force F_{a} with the direction depending on the velocity directing of the platform housing, as well as due to the rolling friction couple M_{rf} with the direction depending on the rotation directing of the wheel. Besides, the significant nonlinearities are due to depending of the F_{a} aerodynamic force on square of the platform velocity, as well as due to depending of the rolling friction couple M_{rf} on the supplied driving couple M_{e}, if the wheel is not rotated. We will use the schematisation of the driving direct current electric

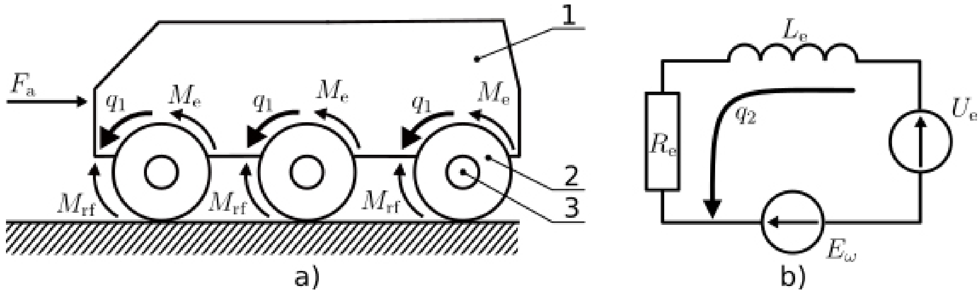

Figure 1: Schematisation of the electromechanical six-wheeled platform (a) with hopusing-1, wheels-2, electric motors-3, as well as schematisation of the equivalent scheme (b) of electric motors

motors like on fig. 1b taking into account the resistance R_e and the inductance L_e of the rotor winding, the supplied power with the voltage $U_e = U_e(t)$, as well as the voltage E_ω generated due to rotation in the magnetic field of the rotor with the electric current in the winding. The state of the electrical part of the driving electric motors will be defined by the generalised coordinate q_2 representing the electric charge in the equivalent electric circuit (fig. 1b) of the rotor winding. We will consider further the motions of the researched wheeled platform starting from the state of rest at the initial time $t = 0$. Taking into account all described here assumptions about schematisation of the researched electromechanical six-wheeled platform, we can use the Lagrange's equation of 2-nd kind and the electromechanical analogies to biuld the differential equations representing the state of this researched wheeled platform, like was in research **?** for example. As the result of these approach, we will have two second ordered ordinary differential equations with the correspondent initial conditions in the following view:

$$J\frac{d^2q_1}{dt^2} = 6B_e\frac{dq_2}{dt} - mg\delta\text{sign}\left(\frac{dq_1}{dt}\right) - kr^3\left(\frac{dq_1}{dt}\right)^2\text{sign}\left(\frac{dq_1}{dt}\right), \tag{21}$$

$$L_e\frac{d^2q_2}{dt^2} = U_e(t) - B_e\frac{dq_1}{dt} - R_e\frac{dq_2}{dt}, \tag{22}$$

$$q_1(0) = 0,\ \frac{dq_1}{dt}(0) = 0, \quad q_2(0) = 0,\ \frac{dq_2}{dt}(0) = 0, \tag{23}$$

where J is the equivalent moment of inertia, and m is the total mass of the researched wheeled platform; δ is the rolling friction coefficient of the wheel; B_e is the electromechanical parameter of the electric motor; k is the coefficient defininmg the aerodynamic forces; r is the equivalent radius of the wheels; $\text{sign}(\cdot)$ is the function defined as the sign of it argument.

To represent the differential equations (21), (22) in adreement with generalised form (1), the following new variables are introduced:

$$x_1 = \frac{dq_1}{dt},\ x_2 = \frac{dq_2}{dt},\ u(t) = U_e(t). \tag{24}$$

The variable x_1 represents the angular velocity of the wheels of the researches electromechanical vehicle (fig. 1a), but the variable x_2 represents the electric current in the electric circuits (fig. 1b) of driving electric motors. Due to these introduced new variables (24), the differential equations (21), (22) will have the following view:

$$J\frac{dx_1}{dt} = 6B_ex_2 - mg\delta\text{sign}(x_1) - kr^3x_1^2\text{sign}(x_1), \quad L_e\frac{dx_2}{dt} = u(t) - B_ex_1 - R_ex_2. \tag{25}$$

It is principally, that first equation (25) is correct only for the case of nonzero velocity $x_1 \neq 0$. To have the differential equation suitable for all values x_1 and agreed with the rolling friction properties, we must introduce the function:

$$M\left(x_1,x_2\right)=\begin{cases}6B_{\mathrm{e}}x_2-mg\delta\operatorname{sign}\left(x_1\right), & x_1\neq 0,\\ \frac{1}{2}\left(6B_{\mathrm{e}}\left|x_2\right|-mg\delta+\left|6B_{\mathrm{e}}\left|x_2\right|-mg\delta\right|\right)\operatorname{sign}\left(x_2\right), & x_1=0.\end{cases} \tag{26}$$

Thus, taking into account the new variables (24), the transformed equation (25) and the introduced function (26), we can represent the inital-value problem (21)-(23) in the following view:

$$\frac{dx_1}{dt}=\frac{M\left(x_1,x_2\right)}{J}-\frac{kr^3}{J}x_1^2\operatorname{sign}\left(x_1\right),\ \frac{dx_2}{dt}=\frac{u\left(t\right)}{L_{\mathrm{e}}}-\frac{B_{\mathrm{e}}}{L_{\mathrm{e}}}x_1-\frac{R_{\mathrm{e}}}{L_{\mathrm{e}}}x_2,\ x_1\left(0\right)=0,\ x_2\left(0\right)=0. \tag{27}$$

It is understandable, that the initial-value problem (27), representing the mathematical model of the researched electromechanical wheeled platform (fig. 1), is actually the particular case of the general representation (1) of the discrete dynamyc systems. The particular views of the state vector $\mathbf{x}$, the control vector $\mathbf{u}$, the vector function $\mathbf{f}\left(t,\mathbf{x};\mathbf{u}\right)$, as well as value t_0 and the vector $\mathbf{x}_0$, corresponded to the particular case (27) of the generalised initial-value problem (1), are fully understandable, so that we will not present them here. At the same time, it is necessary to note, that first differential equation (27) has the significant nonlinearities due to involving the function $\operatorname{sign}\left(\cdot\right)$ and, especially, the function (26), so that linearisation of the problem is not possible in this particular example.

Although, the rotation angle q_1 and the rotation velocity x_1 of the wheels are really the most suitable parameter to represent the state of wheeled platforms, but such parameters are not suitable to research the exploitation processes, so that it is suitable to consider, as example, the following controlled parameter:

$$y=rx_1. \tag{28}$$

The controlled parameter y, defined by the relation (28), represents the velocity of the straight-line motion of the reseaerched wheeled platform (fig. 1a), and this parameter is one of the most important to define the operational modes. Thus, the mathematical model (27) and the controlled parameter definition (28) will allow us considering the optimal controls for changing the straigth-line velocity of the researched electromechanical wheeled platform (fig. 1), and such controls are reduced to finding the correspondent voltages $u\left(t\right)$, which must be supplied on the driving electric motors. As the example of the optimal control problem for the researched electromechanical wheeled platform (fig. 1) represented by the mathematical model (27), (28), we will consider determination of the control $u\left(t\right)$ providing the given straight-line velocity y_f at the given time moment $t=t_f$ after starting from the state of rest at the initial time moment $t=0$, so that the used electric power must be minimal, and the differential equation (11), generally defining the optimality criteria, will have the following particular view:

$$\frac{dJ}{dt}=x_2u\left(t\right),\ J(0)=0. \tag{29}$$

Thus, the mathematical model (26), (27), the controlled parameter (28) and the optimality criteria (29) allow us to consider the correspondent particular optimal control problem for the researched electromechanical wheeled platform (fig. 1).

To find the optimal control $u\left(t\right)$ providing the required the given velocity y_f at the given time t_f moment for the researched electromechanical wheeled platform (fig. 1) starting from the state of rest, we will use the proposed approach based on using the numerical methods. In agreement with the proposed approach, we will represent the control $u\left(t\right)$ as the linear combination of the possible controls corresponded for the different operational modes, and we will use the particular view of the generalised relation (20) as follows:

$$u\left(t\right)=a_1+a_2t, \tag{30}$$

where a_1 and a_2 are the unknown parameters, which must be found to optimise the control.

The parameter a_1, introduced in the control (30), can characterise the steady states operational modes, but the parameter a_2 can characterise the transient operational modes of the researched electromechanical wheeled platform (fig. 1). It is necesdsdary to note also, that the constant voltage, supplied to the driving electric motors in the considered example about the electromechanical wheeled platform (fig. 1), is the sense of the parameter a_1, involved to the control representation (30), due to the last relation (24). So, it is naturally to assume that

$$0 \le a_1 \le a_1^{\max}, \tag{31}$$

where $a_1^{\max}$ is the maximim possible value of the parameter a_1.

Although, the considered control (30) only schematically represents the operational modes of the researched electromechanical wheeled platform, but it is enough to fully illustrate the proposed approach for finding the optimal controls by using the numerical methods. Takingh into account the introduced control (30), the mathematical model (26), (27), (29) and the definition (28) of the controlled parameter for the considered example, we will have the generalised representation of the controlled parameters and the optimality criteria, presented by the relations (16), in the following particular form:

$$y = y\left(t; a_1, a_2\right), J = J\left(t; a_1, a_2\right), \tag{32}$$

where $y\left(t; a_1, a_2\right)$ is the representation of the controlled parameter (28), but $J\left(t; a_1, a_2\right)$ is the representation of the functional defining the optimality criteria, so that all them are biult through solving of the initial-value problem (26), (27) for the control (30) with the given parameters a_1 and a_2.

The relations (32) with the given final values t_f and y_f allows us to define the parameters a_1^{opt} and a_2^{opt} representing the optimal control:

$$a_1^{\text{opt}}, a_2^{\text{opt}}, G\left(a_1^{\text{opt}}, a_2^{\text{opt}}\right) = 0 : F\left(a_1^{\text{opt}}, a_2^{\text{opt}}\right) \le F\left(a_1, a_2\right) \forall a_1, a_2, G\left(a_1, a_2\right) = 0, \tag{33}$$

where $G\left(a_1, a_2\right) = y\left(t_f; a_1, a_2\right) - y_f$ is the function allowing us to define the set of the controls providing ahievement of the given final state; $F\left(a_1, a_2\right) = J\left(t_f; a_1, a_2\right)$ is the function defining the control optimality criteria.

Relation (33) is actually the formulation of the optimisation problem about finding of the minimum of two variable function $F\left(a_1, a_2\right)$ under the restriction defined through the function $G\left(a_1, a_2\right)$. To solve this optimisation problem (33), it is suitable to consider the following functions (fig. 2):

$$G\left(a_1, a_2\right) = a_2 - G_2\left(a_1\right), \quad F_G\left(a_1\right) = F\left(a_1, G_2\left(a_1\right)\right), \tag{34}$$

where $G_2\left(a_1\right)$ is some function chosen so that $G\left(a_1, G_2\left(a_1\right)\right) \equiv 0$.

The introduced functions (34) allow us to reduce the optimisation problem for the two variables function $F\left(a_1, a_2\right)$ with the restriction $G\left(a_1, a_2\right) = 0$ to the more suitable optimisation problem for the one variable function $F_G\left(a_1\right)$:

$$a_1^{\text{opt}} : \quad F_G\left(a_1^{\text{opt}}\right) \le F_G\left(a_1\right) \ \forall a_1, \tag{35}$$

$$a_2^{\text{opt}} = G_2\left(a_1^{\text{opt}}\right), \tag{36}$$

where $0 \le a_1^{\text{opt}} \le a_1^{\max}$ and $0 \le a_1 \le a_1^{\max}$.

The principal difficulties to solve the optimisation problem (35), (36) are in biulding of the functions (34), because of the required functions $G\left(a_1, a_2\right)$ and $F\left(a_1, a_2\right)$ are defined indirectly only through the solution (32) of the initial-value problem (26), (27), (29) taking into account the definition of the controlled parameter (28), the representation (30) of the control and the given final values t_f and y_f. To solve approximately the optimisation problem (35), (36), we will build the discrete representation of first function (34) by means it values defined in the previuosly chosen values (the grid) of the parameter a_1 (fig. 2):

$$a_{1(k)} = (k-1)\,\Delta a_1,\ G_{2(k)} = G_2\left(a_{1(k)}\right), \quad k = 1, 2, \ldots, N, \tag{37}$$

where $a_{1(k)}$ are the chosen values (the grid nodes) of the parameter a_1, but $G_{2(k)}$ are the nodal values of the function $G_2\left(a_1\right)$; N is the chosen number of the wished grid values of the parameter a_1 inside the available interval (31), and $\Delta a_1 = a_1^{\max}/\left(N-1\right)$ is the step of the a_1 parameter grid values.

To find the nodal values $G_{2(k)},\ k = 1, 2, \ldots, N$, we will solve the set of nonlinear implicitly defined equations:

$$G\left(a_{1(k)}, G_{2(k)}\right) = 0, \quad k = 1, 2, \ldots, N. \tag{38}$$

To solve the equations (38), it is possible to use the different numerical methods, but further we will use the Interval Halving (Korn & Korn, 2000). To use this method, it is necessary to compute the values of the function $G\left(a_1, a_2\right)$ implicitly defined in the relation (33) through the nonlinear initial-value problem (26), (27), (29), the controlled parameter definition (28), the control representation (30), as well as the given values t_f, y_f determining the final state. To do it, we will use the corresponded numerical methods (Korn & Korn, 2000) suitable to solve the initial-value problems. The introduced grid nodes and the nodal values (37) allow us to compute the nodal values of second function (34) representing the assumed in this example optimality criteria (fig. 2):

$$F_{G(k)} = F\left(a_{1(k)}, G_{2(k)}\right), \quad k = 1, 2, \ldots, N. \tag{39}$$

To compute the values (39), it is necessary to solve the nonlinear initial-value problem (26), (27), (29) taking into account the controlled parameter definition (28), the control representation (30) as well as the given values t_f, y_f determining the final state. To do it, we will use the corresponded suitable numerical methods (Korn & Korn, 2000). Having the values (39), we can define the integer value k^{opt} corresponded to the optimal control:

$$k^{\text{opt}} \in \{1, 2, \ldots, N\} : F_{G(k^{\text{opt}})} \leq F_{G(k)} \quad \forall k \in \{1, 2, \ldots, N\}. \tag{40}$$

The value k^{opt}, defined by the relation (40), allow us to approximately determine the optimal control (35), (36) as follows:

$$a_1^{\text{opt}} \approx a_{1(k^{\text{opt}})}, \quad a_2^{\text{opt}} \approx G_{2(k^{\text{opt}})}. \tag{41}$$

The approximate values (41) are due to the finite step Δa_1 to represent the possible values (31) of the a_1 parameter, as well as numerical methods applications to compute the nodal values $G_{2(k)}$ and $F_{G(k)}$, $k = 1, 2, \ldots, N$ (fig. 2). So, to have more accurate results (41) for the optimal control, it is necessary to decrease the step Δa_1. Thus, the optimal control is approximately find in the considered example about the electromechanical wheeled platform (fig. 1).

5 Computer simulations for the considered example

To show the opportunities of control optimality researches, we will make the computer simulations of the discussed above electromechanical wheeled platform (fig. 1), represented by the mathematical model including the nonlinear initial-value problem (26), (27), (29), the controlled parameter definition (28), the control representation (30), as well as the given values t_f, y_f determining the final state. To make the computer simulations, we will use the following values of the parameters, involved to the used mathematical model of the researched electromechanical wheeled platform:

$$m = 1500\text{kg},\ J = 35\text{kg}\cdot\text{m}^2,\ r = 0,15\text{m},\ k = 15\text{kg/m},\ \delta = 0,01\text{m},$$

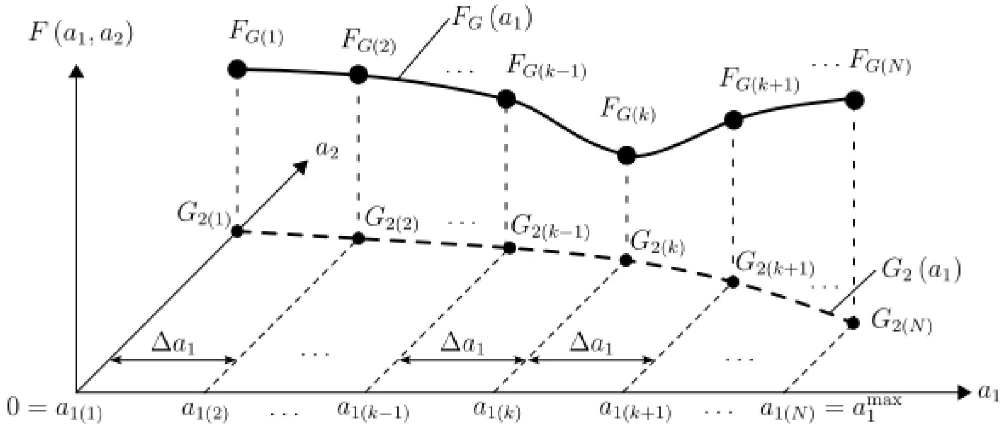

Figure 2: Scheme of computations to approximately find of the optimal control for the researched electromechanical wheeled platform

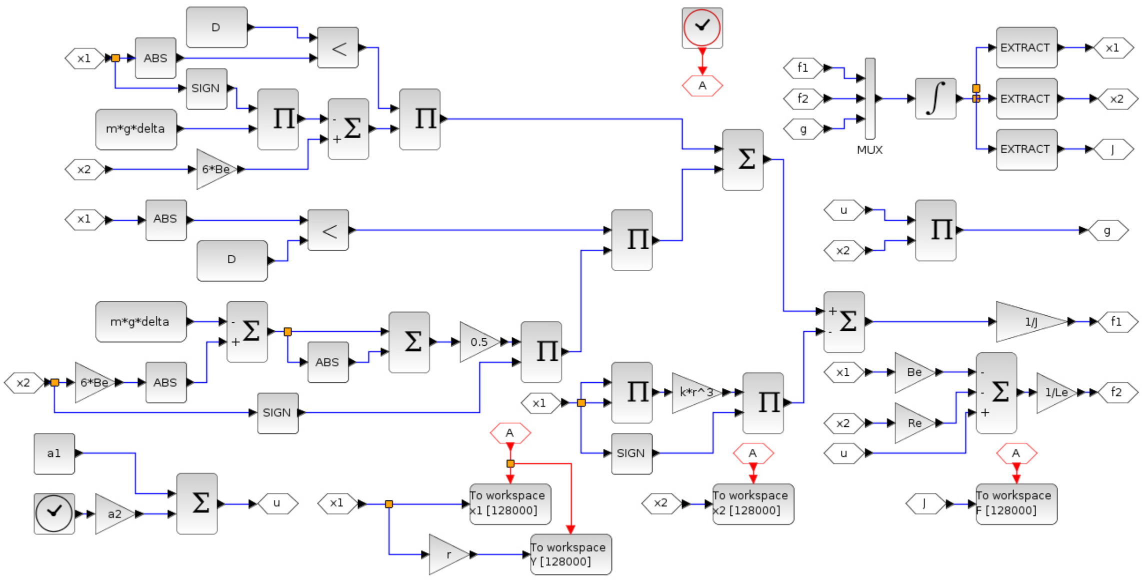

Figure 3: Computer model of the researched electromechanical wheeled platform represented by using the Scilab Xcos tools

$$R_{\mathrm{e}} = 0,22\Omega,\ L_{\mathrm{e}} = 1,5\mathrm{mH},\ B_{\mathrm{e}} = 1,22\mathrm{N}\cdot\mathrm{m/A},\ a_1^{\max} = 12\mathrm{V},\ t_f = 10\mathrm{s}. \tag{42}$$

To do all the computer simulations, we will use the well-known Scilab free open source software.

To represent the researched electromechanical wheeled platform (fig. 1) in agreement with the assumed mathematical model, we will use the computer model developed by using the tools of the Xcos modelling medium envisaged in of the Scilab software, as it is shown on fig. 3. All significant nonlinearities, inherent for the used mathematical model, are took into account in the developed computer model (fig. 3). To have the reliable results, it is necessary to substantiate the correctness of the developed computer model (fig. 3). To do it, we will use the well-known fundamental property, inherent for wheeled platforms, in having the maximum possible velocity corresponded to the equilibrium between the given constant driving and established damping generalised forces at the time moment $t \to \infty$. Due to this fundamental property, we khow, that the solution of the initial-value problem (26), (27), (29), (30) has the follows view:

$$a_2 = 0 \quad \Rightarrow \quad \lim_{t\to\infty} \frac{dx_1}{dt}(t) = 0,\ \lim_{t\to\infty} \frac{dx_2}{dt}(t) = 0,\ \lim_{t\to\infty} x_1(t) = \omega,\ \lim_{t\to\infty} x_2(t) = I, \tag{43}$$

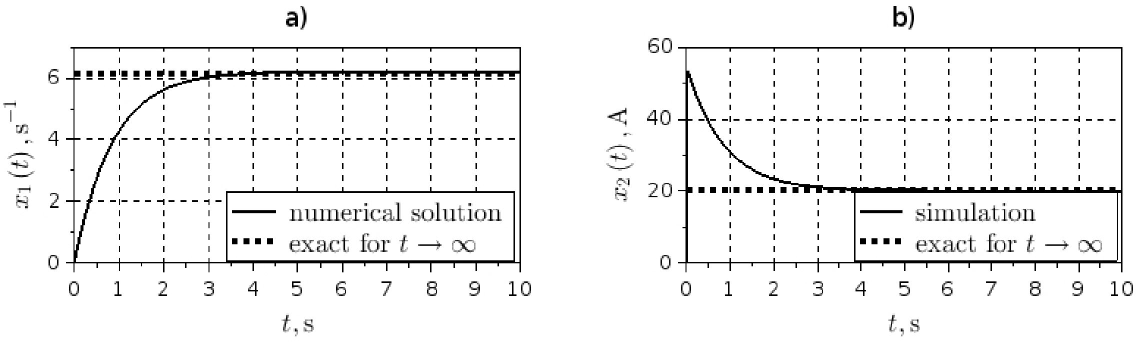

Figure 4: Simulation results for the states of mechanical (a) and electrical (b) parts for the researched electromechanical wheeled platform under in the case $a_1 = 12$V, $a_2 = 0$

where ω and I are some constants reresenting the wheels angular velocity and the electric current in the rotors windings of electric motors in correspondence with the value a_1, which determines the driving genegalised forces.

Taking into account the relations (43), the differential equations (26), (27), (29) for the time $t \to \infty$ will have the following view:

$$6B_{\mathrm{e}}I - mg\delta - kr^3\omega^2 = 0, a_1 - B_{\mathrm{e}}\omega + R_{\mathrm{e}}I = 0, \frac{dJ}{dt} = a_1 I. \tag{44}$$

The equations (44) can be resolved analytically, so that, with the definition (28) of the controlled parameter, we will have the following:

$$I = \frac{B_{\mathrm{e}}\omega - a_1}{R_{\mathrm{e}}}, kr^3\omega^2 - \frac{6B_{\mathrm{e}}^2}{R_{\mathrm{e}}}\omega + \left(mg\delta + \frac{6B_{\mathrm{e}}a_1}{R_{\mathrm{e}}}\right) = 0, J \sim a_1 I t, v = \omega r, \tag{45}$$

where $v \equiv \lim\limits_{t\to\infty} y(t)$.

The relations (45) allow us to define analytically the values ω, I, v, as well as they show us, that the time depending of the value J, defining the optimality criteria, must be linear function for the time $t \to \infty$. So, these relations (45) will be used for benchmarking the results of computer simulations trough the developed model (fig. 3) for the researched electromechanical wheeled platform. On the fig. 4, we can see, that the computer simulations results for the state parameters (the solid curves) of the researched electromechanical wheeled platform are in fully agreement with the properties (43) and with the corresponded exact values (the dot lines) defined analytically through the relations (45). The computer simulation results for the controlled parameter (fig. 5a) of the researched electromechanical wheeled platform are also in fully agreement with the properties (43) and with the corresponded exact values (the dot lines) defined analytically through the relations (45). The computer simulation results (fig. 5a) for the researched electromechanical wheeled platform show us, that the time dependence of the value of the functional, defining the optimality criteria, makes closer to the linear function, exactly as was shown by the correspondent relation (45). Thus, we can assume, that the computer simulations by using the developed model (fig. 3) provide the correct results for the researched electromechanical wheeled platform, so we can use this developed model (fig. 3) further to find the optimal controls.

To find the optimal control for the researched electromechanical wheeled platform, we will build firstly the introduced above function $G_2(a_1)$ through the nodal values (fig. 2). To do it, we will use the grid of the a_1 parameter with the step $\Delta a_1 = 3$V, so that solving of the correspondent nonlinear equations, as was discussed above (fig. 2), leads to the results presented on fig. 6a. We can see (fig. 6a), that the functions $G_2(a_1)$, corresponded to the different y_f values, are the linear functions in the considered particular example. Although, linearities of the functions

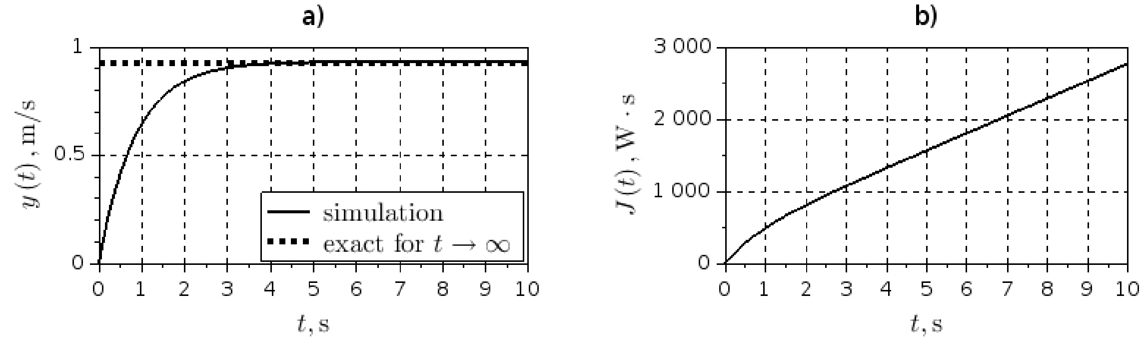

Figure 5: Simulation results for the controlled parameter (a) and the optimality criteria functional (b) for the researched electromechanical wheeled platform in the case $a_1 = 12$V, $a_2 = 0$

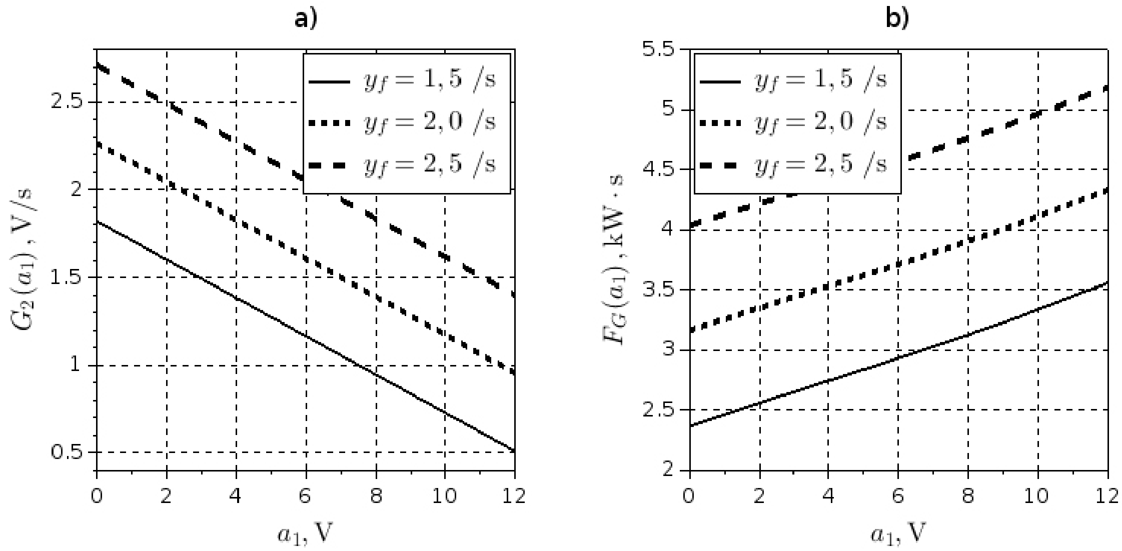

Figure 6: Simulation results about the permissible controls providing the given final state achievement (a) as well as about optimality criteria (b)

$G_2(a_1)$ allow us to simplify the representation of these functions, but it is not principal for the proposed approaches, because this approaches are based on the numerical methods applications independently from the kind of the function $G_2(a_1)$. The built nodal values of the functions $G_2(a_1)$ (fig. 6a) for the different values y_f allow us to find the nodal values of the corresponded functions $F_G(a_1)$, how it was discussed previously (fig. 2). The computation results (fig. 6b) show us, that the functions $F_G(a_1)$, corresponded to the different y_f values, are the linear functions in the considered particular example. Although, linearities of the functions $F_G(a_1)$ allow us to simplify the representation of these functions, but it is not principal for the proposed approaches, because this approaches are based on the numerical methods applications independently from the kind of the function $F_G(a_1)$. So, in any case, the function $F_G(a_1)$, built even through the nodal values, allows us to determine the optimal control, and we can see (fig. 6b), that the optimal controls are corresponded to the values $a_1 = 0$ for the different final values y_f. We can see also (fig. 6b), that implementation of the optimal controls will allow decreasing on about 25% of the energy used to accelerate the electromechanical wheeled platform to the given velocity y_f from the state of the rest during the given time t_f.

To substantiate the value $a_1 = 0$ inherent for the biult optimal controls (fig. 6b), it is necessary to understand the sence of the parameters a_1 and a_2 defining the control (30) of the resesearched electromechanical wheeled platform (fig. 1). It is necessary to remember, that the control (30) in the considered example about the electromechanical wheeled platform (fig. 1) is the electric voltage supplied on the driving electric motors, as it is defined in the used mathematical model (24), (26), (27). So, the involved to the control (30) parameters a_1 and a_2 deteremine the time dependence of the voltage supplied on the driving electric motors of the resesearched electromechanical wheeled platform (fig. 1). The values $a_1 \neq 0$ are corresponded to the controls (30) with the stepwise changing of the voltage, supplied to the driving electic motors of the researched electromechanical wheeled platform (fig. 1), at the initial time moment $t = t_0$. Counterwise, the values $a_1 = 0$ are corresponded to the controls with the smooth changing of the voltage, supplied to the driving electic motors of the researched electromechanical wheeled platform (fig. 1), from the initial zero value at the initial time moment $t = t_0$. In all these cases, the parameter a_2 defines the velocity of smooth changing of the supplied voltage. So, the results $a_1 = 0$, inherent for the optimal controls, show us, that the optimal controls correspond to the voltage smoothly supplied to the driving electric motors of the researched electromechanical wheeled platform. As was previously shown in the article (Alyokhina et al., 2021), the voltage, stepwisely supplied to the driving electric morots, leads to the greater accelerations of the wheeled platform straight motion. It seems naturally, that providing the bigger accelerations requires the bigger supplied power, so it looks really believable, that the optimal controls, corresponding to the minimum used energy, must be without stepwise changing to exclude the bigger accelerations. Thus, the results $a_1 = 0$ for the optimal controls are in the agreement with the necessities of excluding the stepwise controls variations to minimize the accelerations and, possible, the jerks of the wheeled platform to decrease the used energy, providing the wished motions.

6 Conclusions

Although, the theory of optimal control is developed intensively several last decades, but we have no all-conventional approaches for numerical methods application to find directly the optimal controls for the discrete dynamic systems at present. At the same time, such approaches are really required, especially, to consider the applied tasks in different fields. So, the principal reason to make this research is in striving to development of the approaches for numerical methods application suitable to find directly the approximate solutions for the optimal controls of the discrete dinamic systems, and suitable to consider the different applied, but not pure theoretical, tasks. Thus, all the results of this research are in the applied fields of the optimal control.

Researches, striving to development of the universal approaches suitable for the applied tasks about the optimal control, require consideration of the maximum, as possible, generalised mathematical formulations without the particularisations, especially, leading to linear tasks, as well as to patricular optimality criterias, like about the time-optimal. So, the general formulation of the optimal control problem for the discrete dynamic systems is presented similarly as Pontryagin considered it. The principal difference of the considered mathematical formulation comparing with the Pontryagin's consideration is in separate introduction of the controlled parameters in addition to the state and control parameters of the researched system, although, the notion about the controlled parameters is well-known and widely used at present. Separate introduction of the controlled parameters in the optimal control problem formulations is principally required to provide considerations of the applied tasks, especially, because of only some of the parameters, characterising the researched systems, can be principally controlled in different applications usually.

It is shown, that the discrete representation of the controls through the set of the scalar value parameters allows us to reduce the optimal control problem for the discrete dynamic system to consideration of the resolving optimisation problem for the correspondent several variables function with some additional restriction. This several variables function represents the optimality criteria taking into account the mathematical model of the researched system, but this additional restriction needs to have the agreement with the given wished final controlled parameters of the researched system. All these are principaly important, because of it is possible to use a lot of well-known numerical methods to solve directly the resolving optimisation problem for the several variables function with the additional restriction. At the same time, the principal difficulty is in required computation of the values of such optimized several variables function, actually defined implicitly through the initial-value problem representing the mathematical model of the researched systems and the assumed optimality criteria. So, to compute the values of the optimized function, it is necessary to solve the correspondent initial-value problem, and only the numerical methods, like the Runge-Kutta's methods and similars, must be used to do it. Another principal difficulty is in consideration of the additional restriction in the resolving optimisation problem, so that this restriction determines the possible controls providing achievement of the given wished final values for the researched system controlled parameters. Consideration of this additional restriction is reduced to solving the corresponded nonlinear equations defined implicitly through the initial-value problem, representing the mathematical model of the researched system with taking into account the definition of the controlled parameters and theirs given final values. The correspondent well-known numerical methods, like the Interval Halving, the Regula Falsi and others to solve the correspondent initial-value problem, implicitly determining these nonlinear equations, the Runge-Kutta's or others numerical methods must be used. Thus, the proposed approaches not give us the new numerical methods, but they give us the way of using the known numerical nethods to solve approximately the optimal control problem for discrete dynamic systems. The proposed approached cannot claim on solving of the optimal control problem in general, but they are really suitable to consider the particular tasks representing different applications.

The considered particular example about the optimal controls for the electromechanical six-wheeled platform fully presents the typical application technique of the proposed approaches to use the numerical methods for applied tasks consideration. It is shown, that the proposed approaches are not sensitive to the view of the differential equations representing the mathematical model of the researched system, to the view of the optimality criteria, as well as to the definition and to the final values of the controlled parameters. It is also shown, that proposed approaches can be accomplished by means the different universal computer software for scientific computations and computer simulations, so that such different software, having the envisaged standard numerical methods and the scenarios programming tools, can be adopted to find directly the optimal controls for the discrete dynamic systems from different applied fields. On the considered example, the compuiter simulation results allow us to see, that the stepwise control modes must be excluded to provide the optimal control, corresponded to the minimum used energy to accelerate the electromechanical wheeled platform from the state of rest to wished velocity during the wished time. This resulting affirmation can be substantiated by the previously known results, that the stepwise control modes will lead to the bigger accelerations and the bigger jerks of the wheeled platforms motions, requiring the more suplied energy to provide them. So, it looks like the minimization of the used power can be connected with minimization of the motions jerks, at least, on some particular cases. It is also shown, that implementation of the optimal control can decrease the used energy on even 25%.

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